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A 75° Angle Constraint for Plane Minimal T1 Trees

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Abstract. In this paper it is shown that the minimum angle between any 2 edges of an Euclidean plane minimal T1 tree, or 3-size Steiner tree, is at least 75°.

Keywords: Steiner minimal tree, T1 tree, Q-component

Introduction

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ denote a finite collection of n Euclidean plane (regular points). A Steiner minimal tree SMT is a shortest length network interconnecting X_n where (1) all angles between edges are at least 120° and (2) there may be extra points, called Steiner points. It is called full if there are exactly $n-2$ Steiner points and all edges meet at exactly 120°. The underlying graph of a Steiner tree is called the topology. A T1 tree, or 3-size Steiner tree (Du et al., 1991), interconnecting X_n consists of spanning edges and minimal Steiner trees that interconnect 3 regular points, called Q-components.

A minimal (shortest length) T1 tree may contain edges meeting at angles less than 120° and it is conjectured that $L_{\mathrm{SMT}}/L_{\mathrm{T1}} \geq 0.93185\ldots$ where L_{SMT} is the length of a Steiner minimal tree and L_{T1} is the length of a minimal T1 tree. The value 0.93185... may be obtained from 4 points lying as the corners of a square. In this paper it is shown that the angle between any 2 edges of a minimal T1 tree must be least 75°.

The variational approach

For a full discussion see Rubinstein and Thomas (1991). Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be n Euclidean plane points and G and G' be two separate trees each interconnecting X_n with G consisting of k edges. Also suppose G and G' have length L_G and $L_{G'}$ respectively. Let $\rho = L_G/L_{G'}$ be defined as a $\rho = R^k \to R$ function over the domain Δ of the edge lengths of G. If $\inf\rho = \rho_o$, then the first derivative $D_\rho(\boldsymbol{v})$ of ρ in the direction of a vector $\boldsymbol{v}$ is $\frac{\dot{L}_{G'}}{L_G}(\frac{\dot{L}_G}{\dot{L}_{G'}} - \rho_o)$. Thus if $\dot{L}_G < (>)0$ and $\dot{L}_G/\dot{L}_{G'} > (<)0$ then $D_\rho(\boldsymbol{v}) < (>)0$.

Theorem 1. *Let $X_n = \{x_1, x_2, \ldots, x_n\}$ denote a finite collection of n Euclidean plane points and suppose T is a minimal $T1$ tree interconnecting X_n. Then any two edges of T meet at an angle of at least* 75°.

Before proving this theorem the following lemmas will be considered.

Lemma 1. *Let x_2', x_3, x_4', x_5' be the corners of a square such that x_2' lies at the "origin", x_3 lies on the "positive y-axis" and x_5' lies on the "positive x-axis". Let $|\ |$ be the length function. As shown in figure* 1, *let b be the third point of the equilateral triangle $\Delta b x_4' x_5'$, l_U be a line passing through $x_3 x_4'$, I_L, be a line passing through $x_2' x_5'$, l' be a line passing through x_3 and b, C' be a circular arc of radius $|x_2' x_3|$ centered at x_2', p be the intersection point of C' and l', $\Delta x_4 x_5 o$ be an equilateral triangle of side length at most $|x_3 o|$ such that*

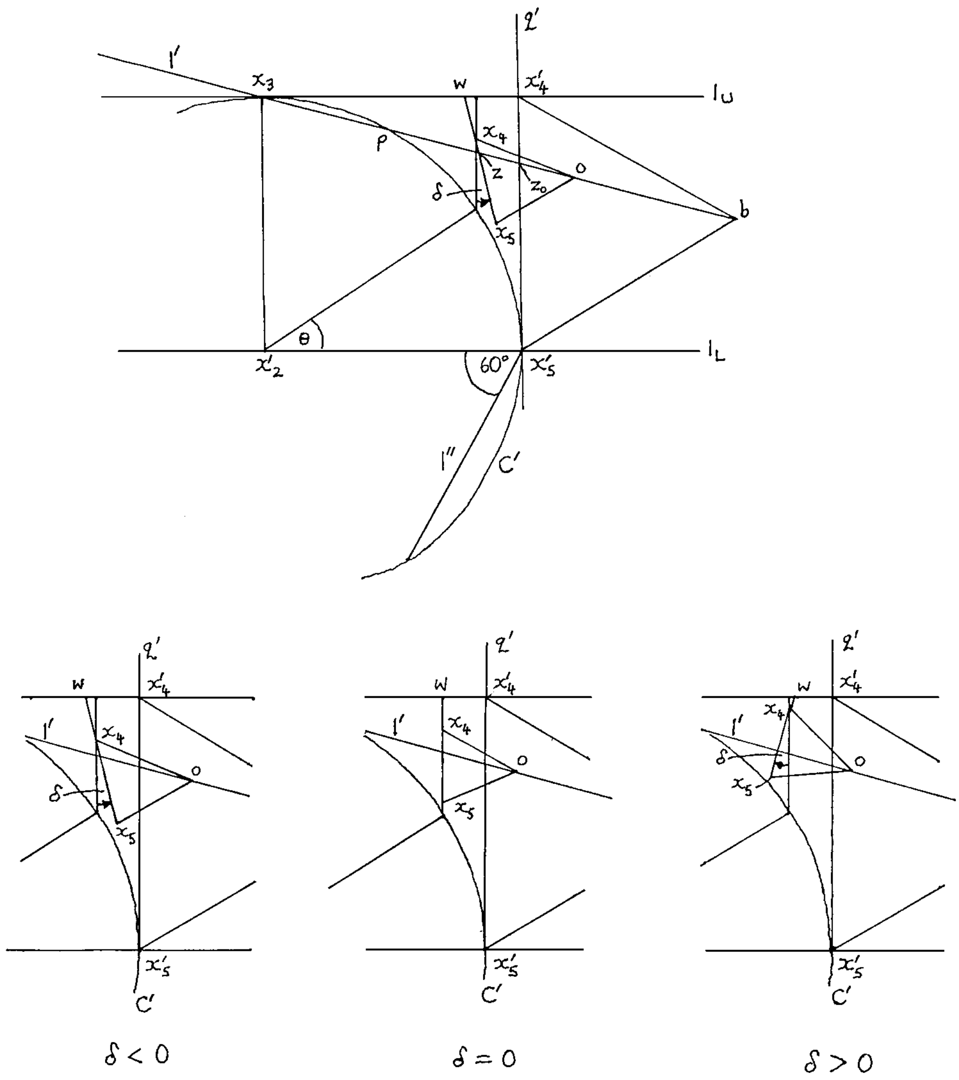


Figure 1. The positioning of $\Delta x_4 x_5 o$.

o lies on l' between b and p, x_4' lies above l', and x_5' lies below l' and outside C'. Also let δ be the angle measured clockwise at x_4 from a line parallel with $x_2'x_3$ to x_4x_5. Then

(i) *if x_4x_5 is parallel to $x_2'x_3(i.e.\ \delta = 0)$ then x_4 lies below l_U.*

(ii) *if x_4 lies above l_U then $\delta \geq 0$.*

Proof: Let q' be a line passing through $x_4'x_5'$.

(i) If x_5 lies on the right hand side of q' then clearly x_4 lies below l_{U}. If x_4 lies on the left hand side of q' then let z_o be the point of intersection of l' and $x_4'x_5'$. Then $|z_o x_5'|/|x_4'x_5'| = \cos 30^\circ \cdot \tan 15^\circ + 0.5$. It is only necessary to consider when x_5 lies on C'. Let z be the intersection point of l' and x_4x_5, w be the intersection point of l_{U} and a line passing through x_4 and x_5. If $\theta = \angle x_5'x_2'x_5$, then

$$|zx_5| = |x_2'x_3|(1 - \sin\theta - \cos\theta \cdot \tan 15^\circ), \qquad |wx_5| = |x_2'x_3|(1 - \sin\theta),$$

and

$$\frac{|zx_5|}{|wx_5|} = 1 - (\cos\theta \cdot \tan 15^\circ)/(1 - \sin\theta) \leq \left(\sqrt{3/2}\right) \cdot \tan 15^\circ + 0.5 = \frac{|z_o x_5'|}{|x_4'x_5'|}$$

for $0^\circ \leq \theta \leq 60^\circ$. Thus X_4 must lie below l_{U}. Note that at $\theta = 60^\circ$, $p = x_5$.

(ii) If x_4 lies on the right hand side of q' then $\delta > 0$. If x_4 lies on the left hand side of q' then suppose x_4 lies above l_{U} and $\delta < 0$. It will be shown that x_5 lies in C'. Note that it is only necessary to consider when x_4 lies on l_{U}. By definition o lies on l'. Thus if $o = b$, then x_5 will lie inside C' on a line, l'', 60° to l_{L} at x_5'. (When $x_4 = x_3$ or $x_4 = x_4'$, x_5 lies on C'). As o moves along l' toward p, x_5 will clearly remain inside C'. When x_5x_4 is parallel to $x_2'x_3$, i.e. corresponding to when $\delta = 0$, x_5 will still lie inside as a consequence of (i). Thus δ cannot be less than zero. □

Lemma 2. *Suppose $X_4 = \{x_2, x_3, x_4, x_5\}$ is a set of four Euclidean plane points connected by a minimal T1 tree G consisting of the spanning edge x_2x_3 and the full Q-component $Q(x_3, x_4, x_5)$. Then the angle θ between x_2x_3 and $Q(x_3, x_4, x_5)$ at x_3 is at least 75°.*

Proof: The aim will be to show that a shorter tree than G exists by replacing one or both components with different components of shorter sum total length. Suppose x_2x_3 and $Q(x_3, x_4, x_5)$ meet at x_3 at an angle of $\theta < 75^\circ$. Let s be the Steiner point of $Q(x_3, x_4, x_5)$ and o be the third point of the equilateral triangle Δox_4x_5. Then $|ox_3| = |x_3s| + |x_4s| + |x_5s|$ (Melzak, 1961). Now, as defined in Lemma l, let x_2', x_4', x_5' and b be points such that x_2', x_3, x_4', x_5' form the corners of a square and x_4', x_5', b form the corners of an equilateral triangle so that b co-incides with o. Note that $\angle bx_3x_2' = 75^\circ$. Let C be a circular arc of radius $|x_3b|$ centered at b, and define $l_{\mathrm{U}}, l_{\mathrm{L}}, l', C'$, and δ as in Lemma 1. There are a number of situations to consider.

(a) x_2 lies strictly inside the curve C and x_2b does not intersect x_4x_5. Let u be the intersection point of x_2x_3 and a line passing through x_5 and o (figure 2). Clearly $|x_3o| > |uo|$

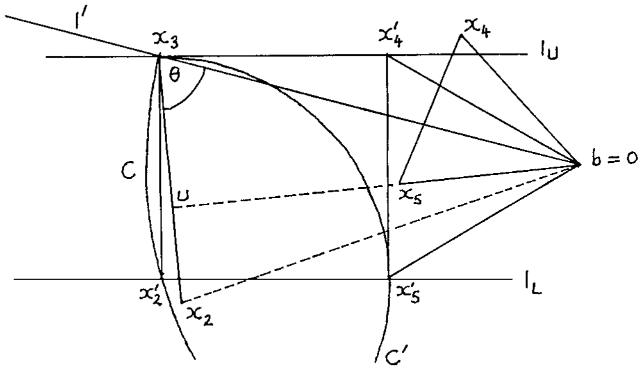


Figure 2. The positioning of x_2.

so $L_G = |x_2x_3| + |Q(x_3, x_4, x_5)| > |x_2x_3| + |uo| = (|x_2x_3| + |ux_5|) + |x_5o| \geq |Q(x_2, x_3, x_5)| + |x_4x_5|$. Thus G is not minimal as $\{Q(x_2, x_3, x_5), x_4x_5\}$ is shorter. Note that $Q(x_2, x_3, x_5)$ may or may not be full.

(b) x_2 lies strictly inside the curve C and does intersect x_4x_5. In this case $|Q(x_3, x_4, x_5)| = |x_3o| > |x_2o| = |Q(x_2, x_4, x_5)|$. Therefore $Q(x_3, x_4, x_5)$ may be replaced by the shorter $Q(x_2, x_4, x_5)$ which again may or may not be full.

(c) x_2 lies on or outside the curve C. Note that x_2 must lie strictly below the line l_L else $\theta \geq 75°$. If x_5 lies inside the curve C' then x_2x_3 may be replaced by the shorter x_2x_5. This will be apparent as x_5 will strictly be contained in the circular arc of radius x_2x_3 centered at x_2.

Now suppose x_5 does not lie inside C'. If $\delta < 0$ then by Lemma 1, x_4 must lie strictly below l_U.

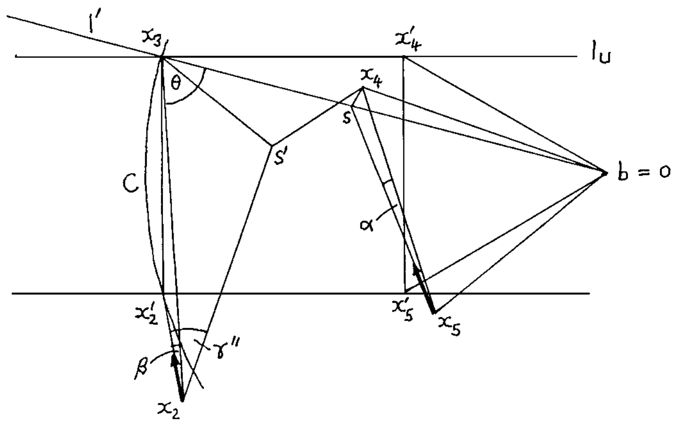
Consider $G' = \{x_4x_5, Q(x_2, x_3, x_4)\}$ (figure 3a). Let s' be the Steiner point of $Q(x_2, x_3, x_4)$, $0 < \beta = \angle x_2', x_2, x_3, \beta < \gamma'' = \angle x_2', x_2, s'$ and assume $\rho' L_G / L_{G'} \leq 1$. Move x_2 toward x_2' such that $x_2'x_2$ decreases at a rate of -1. The $L_G = -\cos\beta < -\cos\gamma'' = \dot{L}_{G'} < 0$ so $\dot{L}_G/\dot{L}_{G'} > 1$ and $D_{\rho'}(\mathbf{v}) < 0$. Thus the ratio ρ' will decrease. Similarly if $\alpha = \angle s, x_5, x_4$ then move x_5 toward s so that $\dot{L}_G = -1 < -\cos\alpha = \dot{L}'_G < 0$ to again give $\dot{L}_G/\dot{L}_{G'} > 1$ and $D_{\rho'}(\mathbf{v}) < 0$. Note too that as b is fixed, o will move along l' toward x_3 and δ will increase. When $x_2 = x_2'$ and $\delta = 0$, x_3 and x_2 lie on l_U and l_L respectively and x_4 and x_5 both lie between l_U and l_L. Note here that x_5 may or may not now lie inside C''.

Let w_2, w_3, x_4, x_5 be the four points of a rectangle such that w_3 lies on l'. (figure 3b) and note that as x_4 lies strictly below l_U, $w_3 \neq x_3$.

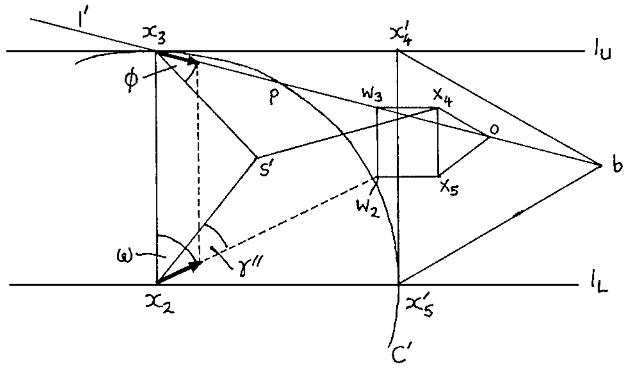
Move x_3 toward w_3 along l' and x_2 toward w_2 at such rates so that x_2x_3 is always perpendicular to l_U and l_L. Let $\varnothing = \angle w_3, x_3, s'$ and $\omega = \angle w_2, x_2, x_3$. Then $\dot{L}_G = -1 - \cos 75° - \sin 75° \cdot \cot\omega$, and $\dot{L}_{G'} = -\cos\varnothing - \sin 75° \cdot \operatorname{cosec}\omega \cdot \cos(\omega - \varnothing + 15°)$.

Consider $f(\omega, \varnothing) = -\dot{L}_G + L_{G'}$. Then

$$f(\omega, \varnothing) = 1 + \cos 75° - \cos\varnothing + \sin 75° \cdot \left\{ \frac{(\cos\omega \cdot \cos(\omega - \varnothing + 15°))}{\sin\omega} \right\}$$



(a)



(b)

Figure 3. Regular point movement toward rectangular configuration.

and

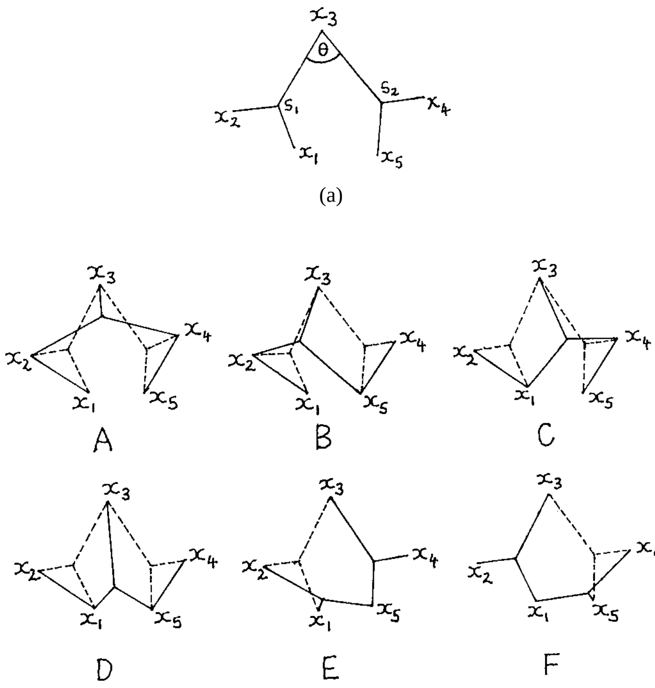
$$\delta f(\omega, \varnothing)/\delta\omega = -\sin 75^\circ \cdot \left\{ \frac{(1 - \cos(15^\circ - \varnothing))}{\sin^2 \omega} \right\} \leq 0$$

for $0 < \omega \leq 90^\circ$.

Thus $f(\omega, \varnothing) \geq f(90, \varnothing) = 1 + \cos 75^\circ - \cos \varnothing - \sin 75^\circ \cdot \cos(105^\circ - \varnothing)$. The unique minimum to this equation, for $0^\circ < \varnothing < 60^\circ$ occurs when $\varnothing = 51.206..^\circ$

So $f(\omega, \varnothing) \geq f(90^\circ, 51.206..) = 0.0617339... > 0$ giving $\dot{L}_G/\dot{L}_{G'} > 1$ and a decreasing ρ'. When $w_2 = x_2$ and $w_3 = x_3$, L_G and $L_{G'}$ may be calculated directly and both have the same value. Therefore a contradiction arises as L_G cannot be shorter than $L_{G'}$.

If $\delta \geq 0$ then x_5 must lie above l_L else $\delta < 0$ or x_5 lies inside C'. Consider $G' = \{x_4x_5, Q(x_2, x_3, x_5)\}$ and assume $\rho' = L_G/L_{G'} \leq 1$. Let s' be the Steiner point of $Q(x_2, x_3,$



(a)

(b)

Figure 4. Defined collection of 6 T1 trees.

x_5). It is now possible to follow a similar procedure as was used in the previous situation when $\delta < 0$ to arrive at the same contradiction that L_G cannot be shorter than $L_{G'}$. □

Definition 1. Suppose $X_5 = \{x_1, x_2, x_3, x_4, x_5\}$, a collection of 5 Euclidean plane points, is interconnected by a T1 tree G consisting of two full Q-components $Q(x_1, x_2, x_3)$ and $Q(x_3, x_4, x_5)$ with Steiner points s_1 and s_2 respectively (figure 4a). Then define $M = \{A, B, C, D, E, F\}$ to be the set of six T1 trees (figure 4b) as follows.

$$A = \{x_1x_2,\ Q(x_2, x_3, x_4),\ x_4x_5\}$$
$$B = \{x_1x_2,\ Q(x_2, x_3, x_5),\ x_4x_5\}$$
$$C = \{x_1x_2,\ Q(x_1, x_3, x_4),\ x_4x_5\}$$
$$D = \{x_1x_2,\ Q(x_1, x_3, x_5),\ x_4x_5\}$$
$$E = \{Q(x_1, x_2, x_5),\ Q(x_3, x_4, x_5)\}$$
$$F = \{Q(x_1, x_2, x_3),\ Q(x_1, x_4, x_5)\}$$

Definition 2. Suppose $X_5 = \{x_1, x_2, x_3, x_4, x_5\}$, a collection of five Euclidean plane points, is interconnected by a T1 tree G consisting of two full Q-components $Q(x_1, x_2, x_3)$ and

$Q(x_3, x_4, x_5)$, with Steiner points S_1 and S_2 respectively. Define Δ to be the configuration space consisting of the six non negative edge lengths of G. i.e. $\Delta = \{s_1x_1, s_1x_2, s_1x_3, s_2x_3, s_2x_4, s_2x_5\}$ such that the sum of the lengths is equal to 1 and the angle between s_1x_3 and s_2x_3 is fixed and is at most 75°.

Lemma 3. *The length of any $T1$ tree with respect to Δ is a convex function.*

Proof: The general result is proved by Du et al. (1991). (As all the angles and the topology of G are fixed, a point of $\Delta \subset R^5$ will determine the configuration of the regular points. The length of any component of a T1 network interconnecting G can then be written as a vector sum and its length shown to be a convex function.) □

Lemma 4. *Suppose $X_5 = \{x_1, x_2, x_3, x_4, x_5\}$ is a collection of five Euclidean plane points lying in some configuration such that*

(1) *The T1 tree G (with length L_G) interconnecting X_5 consisting of two full Q-components $Q(x_1, x_2, x_3)$ and $Q(x_3, x_4, x_5)$ exists and is minimal*;

(2) *The angle θ between the two edges s_1x_3 and s_2x_3 is strictly less than 75°.*

Suppose the maximum value of $L_G/L_{G'}$ for all $G' \in M$ at the configuration is ρ^*. Then there exists another configuration of X_5 such that G, consisting of the two full Q-components $Q(x_1, x_2, x_3)$ and $Q(x_3, x_4, x_5)$, exists, $\theta = 75^\circ$, and $L_G/L_{G'} \leq \rho^*$.

Proof: As G is assumed to be minimal at the initial configuration, $\rho^* \leq 1$.

Note that if a line x_3x_1 intersects x_5s_2 then G cannot be minimal as $Q(x_1, x_2, x_3)$ may be replaced by a shorter $Q(x_1, x_2, x_5)$ giving $L_G/L_{G'} > 1$ with $G' = E \in M$. Similarly if a line x_3x_5 intersects s_1x_1 then G cannot be minimal as $Q(x_3, x_4, x_5)$ may be replaced by a shorter $Q(x_1, x_4, x_5)$ giving $L_G/L_{G'} > 1$ with $G' = F \in M$.

Thus suppose x_3x_1 does not intersect x_5s_2 and x_3x_5 does not intersect s_1x_1. L_G remains constant if $Q(x_1, x_2, x_3)$ is pivoted about x_3 to increase θ and $L_{G'}$ does not decrease for any $G' \in M$. Thus $L_G/L_{G'}$ will decrease for any $G' \in M$ and so $L_G/L_{G'} \leq \rho^*$ for any $G' \in M$. □

Remark. If $L_{G'}$ remains constant for some $G' \in M$ when θ is increased, then $L_G/L_{G'} < 1$. This can be seen for each $G' \in M$. For example, suppose $G' = A \in M$. If $Q(x_2, x_3, x_4)$ is full then $L_{G'}$ must strictly increase. If $Q(x_2, x_3, x_4) = \{x_2x_3, x_3x_4\}$ then $L_{G'}$ remains constant. However in this situation it is obvious that $L_G/L_{G'} < 1$. Note that $Q(x_2, x_3, x_4)$ cannot be $\{x_2x_3, x_2x_4\}$ or $\{x_3x_4, x_2x_4\}$. Thus, by similar consideration of all $G' \in M$, it will be clear that when $\theta = 75^\circ$, $L_G/L_{G'} < 1$ for all $G' \in M$.

Proof of Theorem 1: Consider two edges of T meeting at a common vertex with the angle between them strictly less than 75°. By definition each edge must either be a spanning edge or belong to some full Q-component. Let the two components be denoted by G. The proof will aim to show that a shorter network may be obtained by removing one or both components of G and by replacing them with different components of shorter sum length. There are 3 cases to consider. □

First Case. G consists of two spanning edges. Clearly the edges may be replaced by a full Q-component of strictly shorter length.

Second Case. G consists of one spanning edge and one full Q-component. By Lemma 2 there exist components of strictly shorter sum length which may be used as replacements.

Third Case. G consists of two full Q-components. In this case, the proof follows the ideas of Du and Hwang (1992).

Without loss of generality suppose G interconnects $X_5 = \{x_1, x_2, x_3, x_4, x_5\}$ and G consists of the two Q-components $Q(x_1, x_2, x_3)$ and $Q(x_3, x_4, x_5)$ with the angle θ between s_1x_3 and s_2x_3 (s_1, s_2 the Steiner points of $Q(x_1, x_2, x_3)$ and $Q(x_3, x_4, x_5)$ respectively) at x_3 strictly less than 75°. Note that as T is minimal G must also be minimal. It is now only necessary to restrict attention to G. Consider $G' \in M$. Then $L_G/L_{G'} \leq 1$ and by the Remark following Lemma 4 a minimum value of $L_G/L_{G'}$ strictly less than 1 for all $G' \in M$ will occur for some configuration of X_5 when $\theta = 75°$.

Consider the configuration space Δ as defined in Definition 2 and set $\theta = 75°$. Then there exists some interior point $p_o \in \Delta$ such that G is a minimal T1 tree with $|x_3s_1| \geq |s_2x_3| > 0$ (figure 5b).

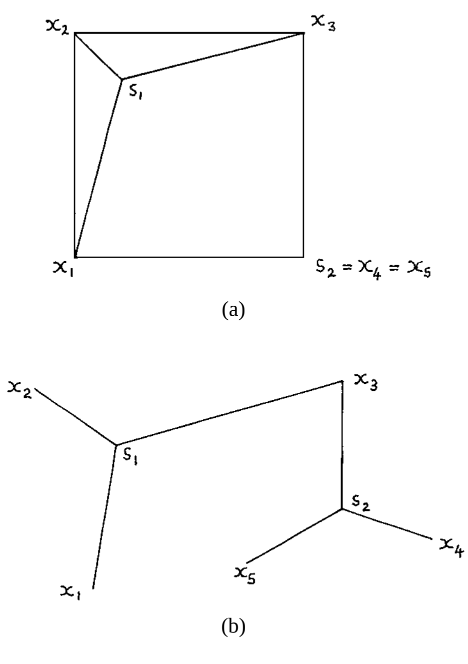


Figure 5. a) Square configuration of regular points, b) $|x_3s_1| \geq |s_2x_3| > 0$.

Let p_1 correspond to a configuration of X_5 such that

$$|s_2x_4| = |s_2x_5| = 0$$

$$\frac{|s_2x_3|}{|x_3x_4|} = \frac{1}{\left(1 + \frac{2}{\sqrt{3}}(\sqrt{2} + \sin 15^\circ)\right)},$$

$$\frac{|s_1x_2|}{|x_3x_4|} = \frac{(2\sin 15^\circ)}{\left(\sqrt{3}\left(1 + \frac{2}{\sqrt{3}}(\sqrt{2} + \sin 15^\circ)\right)\right)},$$

and

$$\frac{|s_1x_1|}{|x_3x_4|} = \frac{|s_1x_3|}{|x_3x_4|} = \frac{\sqrt{2}}{\left(\sqrt{3}\left(1 + \frac{2}{\sqrt{3}}(\sqrt{2}\sin 15^\circ)\right)\right)}$$

i.e. when $s_2 = x_4 = x_5$ and x_1, x_2, x_3, and $x_4(=x_5)$ lie as the corners of a square (figure 5a). Note that for any $G' \in M$, $L_G/L_{G'} = 1$ at p_1.

Now consider the path $(1-\lambda)p_1 + \lambda p_o \in \Delta$ for $\lambda \geq 0$. Note that since $|s_2x_4| = |s_2x_5| = 0$ at p_1 but $|s_2x_4| > 0$ and $|s_2x_5| > 0$ at p_o, some edge of G must decrease at a constant rate as λ increases. Let $r = |s_1x_3|/|s_2x_3|$. Then $r(\lambda)$ is an increasing function. (At p_1, $r = \sin 45^\circ/\sin 60^\circ < 1$, and at $p_o, r \geq 1$). Thus there will exist some $\lambda > 1$ such that one of the following occurs.

(a) G intersects itself.
(b) $|s_1x_3| \cdot 2\cos 75^\circ \leq |s_2x_3| \leq |s_1x_3|$ and at least one Q-component of G is not full.
(c) $|s_2x_3| = |s_1x_3| \cdot 2\cos 75^\circ$ and both Q-components of G are full.
(d) $|s_1s_3| \cdot 2\cos 75^\circ > |s_2x_3|$

Let $p_1' \in \Delta$ correspond to the smallest $\lambda > 1$ for which one of the above occurs. Then to prove Theorem 1 it is necessary to find a T1 tree $G' \neq G$ such that $L_G/L_{G'} \geq 1$ at both p_1 and $p_1' \in \Delta$. It will then follow that $L_G/L_{G'} \geq 1$ at p_o by the convexity of $L_{G'}$ and hence a contradiction that G is minimal.

Each situation is considered separately.

(a) G intersects itself. There are two possibilities.

If s_2x_5 intersects s_1x_1 (figure 6a) then choose $G' = \{Q(x_1, x_2, x_5), Q(x_3, x_4, x_5)\} = E \in M$. Since $|Q(x_1, x_2, x_5)| \leq |Q(x_1, x_2, x_3)|$ at p_1', $L_G/L_{G'} \geq 1$, at both p_1 and p_1'. Thus $L_G/L_{G'} \geq 1$ at p_o by the convexity of G' and so contradicting the minimality of G.

If s_1x_1 intersects s_2x_5 (figure 6b) then choose $G' = \{Q(x_1, x_2, x_3), Q(x_1, x_4, x_5)\} = F \in M$ and a similar argument follows.

(b) $|s_1x_3| \cdot 2\cos 75^\circ \leq |s_2x_3| \leq |s_1x_3|$ and at least one Q-component of G is not full. In this case p_1' lies on the boundary of Δ. Assume G does not intersect itself. Note that both x_4s_2 and x_5x_2 have non zero length at p_1', and $|s_1x_3| \geq |x_3s_2| > 0$ also at p_1'. Thus only x_1s_1 or x_2s_1 can have zero length at p_1'.

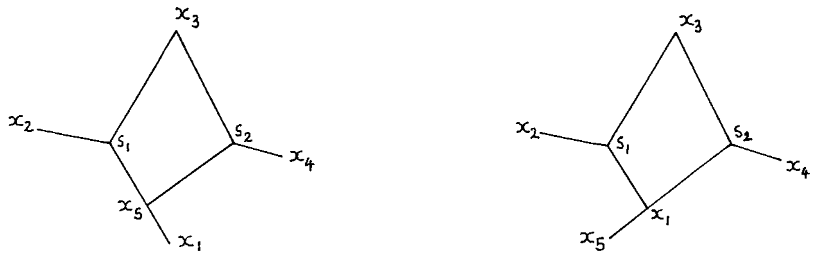


Figure 6. a) s_2x_5 intersecting s_1x_1, b) s_1x_1 intersecting s_2x_5.

The problem can now be considered as a geometric problem. Let x_4', x_5' be points such that s_1, x_3, x_4', x_5' form the corners of a square. Also let o' be the third point of the equilateral triangle $\Delta o's_1x_3$, C' be the circular arc of radius $|s_1x_3|$ at S_1, o be the third point of the equilateral triangle Δox_4x_5, q be a line passing through o' and the midpoint of $x_4'x_5'$, q' be a line passing through o' and S_1, l_U be a line passing through x_3 and x_4', l_L. be a line passing through s_1x_5', and define δ be the angle measured clockwise at x_4 from a line parallel with $x_4'x_5'$ to x_4x_5. Note that if $|s_2x_3| = |s_1x_3| \cdot 2\cos 75^\circ$ then s_2 will lie on C' (figure 7).

There are two situations to consider.

(b.1) If x_5 lies inside C' then $|s_1x_5| \leq |s_1x_3|$ and $|Q(x_1, x_2, x_5)| \leq |Q(x_1, x_2, x_3)|$. Thus G' can be chosen to be $E \in M$ to give $L_G/L_{G'} \geq 1$ and a contradiction that G is minimal at p_o.

(b.2) If x_5 lies outside C' then there are a number of situations to consider.

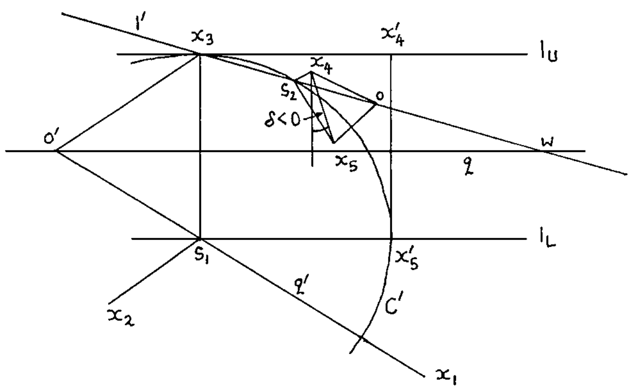


Figure 7. The location of x_5.

(i) x_4 lies below l_U and x_5 lies above l_L. If $|x_1s_1| = 0$ and $\delta \le 0$ then choose G' to be $\{x_1x_2, Q(x_1, x_3, x_4), x_4x_5\} = C \in M$. If x_4 lies strictly below l_U then follow similarly the procedure used in Lemma 2 to show that $|s_1x_3| + |Q(x_3, x_4, x_5)| \ge |x_4x_5| + |Q(x_1, x_3, x_4)|$ to give the contradiction. If x_4 lies l_U then it may be shown again that $|s_1x_3| + |Q(x_3, x_4, x_5)| \ge |x_4x_5| + |Q(x_1, x_3, x_4)|$ by decreasing the edges $s_1x_3 = x_1x_3$ at $s_1 = x_1$ or decreasing s_2x_5 at x_5 until x_1, x_3, x_4, x_5 lie as the corners of a rectangle from which $L_G = L_{G'}$ to give the contradiction of the minimality of G.

Similarly if

$|x_2s_1| = 0$ and $\delta \le 0$ then choose G' to be $\{x_1x_2, Q(x_2, x_3, x_4), x_4x_5\} = A \in M$.
$|x_1s_1| = 0$ and $\delta > 0$ then choose G' to be $\{x_1, x_2, Q(x_1, x_3, x_5), x_4x_5\} = D \in M$.
$|x_2s_1| = 0$ and $\delta > 0$ then choose G' to be $\{x_1x_2, Q(x_2, x_3, x_5), x_4x_5\} = B \in M$.

(ii) x_4 lies above l_U and x_5 lies above l_L. In this situation $\delta > 0$ so if $|x_1s_1| = 0$ choose G' to be $D \in M$, and if $|x_2s_1| = 0$ choose G' to be $B \in M$ and proceed as in (i).

(iii) x_4 lies below l_U and x_5 lies below l_L. If $\delta \le 0$ and $|x_1s_1| = 0$ choose G' to be $C \in M$, and if $\delta \le 0$ and $|x_2s_1| = 0$ choose G' to be $A \in M$. Move x_5 toward s_2 to decrease $L_G/L_{G'}$ until $\delta = 0$. If x_5 lies above l_L when $\delta = 0$ then proceed as in Lemma 2 to obtain the contradiction. If x_5 lies below l_L when $\delta = 0$ then it is not possible to proceed as in Lemma 2. It will therefore be necessary to re-examine the problem. Note here that o lies below q. This situation will now be examined in (iv).

If $\delta > 0$ then o must lie below q. This situation will also now be examined in (iv).

(iv) x_4 lies above l_U and x_5 lies below l_L. In this situation o lies below q so $|x_3o| \ge |s_1o|$.

First suppose $|x_1s_1| = 0$, (figure 8a). If s_1o intersects x_4x_5 then $|Q(x_1, x_4, x_5)| \le |Q(x_3, x_4, x_5)|$ so choose G' to be $F \in M$. If s_1o does not intersect x_4x_5 then note that o must also lie below l_L. Let u be the point of intersection of s_1x_3 and a line passing through ox_5. Then clearly $|x_3o| > |uo| \ge |x_1x_5| +$

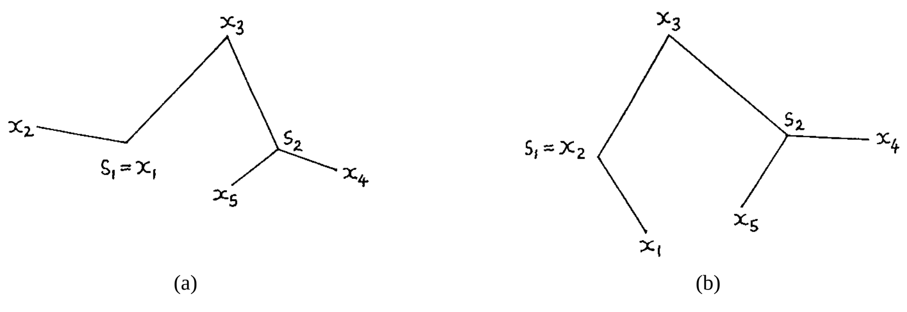


Figure 8. a) $|x_1s_1| = 0$, b) $|x_2s_1| = 0$.

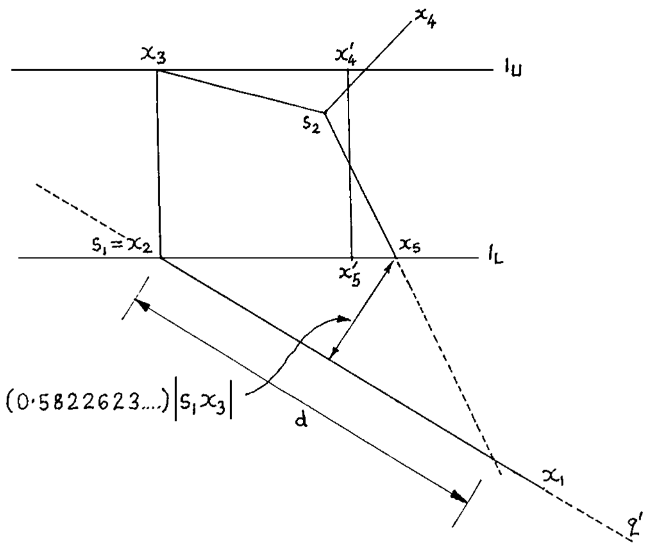


Figure 9. x_1o does not intersect x_4x_5.

$|x_5o|$. Thus $Q(x_3, x_4, x_5)$ may again be replaced by the shorter $Q(x_1, x_4, x_5)$ and G' may be chosen to be $F \in M$.

Next, suppose $|x_2s_1| = 0$, (figure 8b). If x_1o intersects x_4x_5 then $|Q(x_1, x_4, x_5)| \leq |s_1o| \leq |Q(x_3, x_4, x_5)|$ by geometric considerations so G' may again be chosen to be $F \in M$.

Now suppose x_1o does not intersect x_4x_5. Define d to be the distance from s_1 to the intersection point between q' and a line passing through s_2 and x_5 when $|x_3s_2| = |x_3s_1|$, (figure 9).

If $|x_1s_1| \geq d = (\cos 52.5°/\cos 67.5°) \cdot |x_3s_1| = (1.5907703...).\,|x_3s_1|$ (i.e. in this situation x_5 must lie above q'.), then the shortest distance between x_5 and x_1s_1 is at most $d \cdot \tan 30°.|x_3s_1|/(\tan 30° + 1) = (0.5822623...).\,|x_3s_1| < |x_3s_1|$. Thus $|Q(x_1, x_2, x_3)| \geq |Q(x_1, x_2, x_5)|$ so G' may be chosen to be $E \in M$.

Now suppose $x_1s_1 < d$. If x_5 lies below q' then o must also lie below q'. Let u be the point of intersection of x_3s_1 and a line through o and x_5. Then clearly $|x_3o| > |uo| \geq |s_1x_5| + |x_5o| \geq |x_1x_5| + |x_5o|$. Thus $Q(x_3, x_4, x_5)$ may again be replaced by the shorter $Q(x_1, x_4, x_5)$ and G' may be chosen to be $F \in M$.

If x_5 lies above q' then there are two situations.

If $|x_1x_5| \leq |x_3s_1|$ then replace $Q(x_1, x_2, x_3)$ by the shorter $Q(x_1, x_2, x_5)$ and choose G' to be $E \in M$.

If $|x_1x_5| > |x_3s_1|$ then $|x_1s_1|$ can be at most $(d - |x_3s_1|) = (0.5907703...).|x_3s_1|$.

Note that the angle between x_1x_5 and q' at x_1 is at most the angle that $x_5'x_1$ is with q' and is less than 90°, (figure 10). Let u' be the intersection point between x_1s_1 and

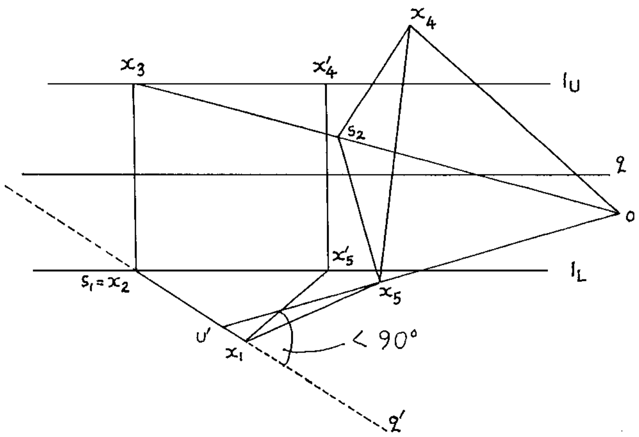


Figure 10. The angle between x_1x_5 and q' at x_1.

a line passing through o and x_5. Then the angle between $u'o$ and q' is also less than 90° and $|x_3o| > |u'o| = |u'x_5| + |x_5o| > |x_1x_5| + |x_5o|$. Therefore $Q(x_3, x_4, x_5)$ may be replaced by the shorter $Q(x_1, x_4, x_5)$ and G' may be chosen to be $F \in M$. Similarly if u' is the intersection point of x_3s_1 and a line passing through o and x_5 then $|x_3o| > |u'o| \geq |s_1x_5| + |x_5o| \geq |x_1x_5| + |x_5o|$ and G' may be chosen to be $F \in M$ again.

(c) $|s_2x_3| = |s_1x_3| \cdot 2\cos 75^\circ$ and both Q-components of G are full.

If x_5 lies inside C' then $|Q(x_1, x_2, x_5)| \leq |Q(x_1, x_2, x_3)|$ so G' may be chosen to be $E \in M$.

If x_5 lies outside C' then as s_2 lies on C', x_5 lies below l_L and $|s_2x_2| \geq \sqrt{2} \cdot |s_1x_3|$. Let w be the intersection point of q and a line passing through x_3 and s_2, (figure 7). Then $|x_3o| = |Q(x_3, x_4, x_5)| = |s_2x_3| + |s_2x_4| + |s_2x_5| > (2\cos 75^\circ + \sqrt{2}) \cdot |s_1x_3| > |x_3w|$. Thus o must lie below q. The proof may now follow similarly to that of (b.iv).

(d) $|s_1x_3| \cdot 2\cos 75^\circ > |s_2x_3|$.

If $|s_1x_3| \cdot 2\cos 75^\circ < |s_2x_3|$ at p_0 then p_1' can always be chosen to satisfy one of the conditions (a), (b), or (c) for some $\lambda > 1$. If at p_o, $|s_1x_3| \cdot 2\cos 75^\circ \geq |s_2x_3|$, then it only needs to be shown that G cannot be minimal at p_0 for $\theta \leq 75^\circ$. Note that $|s_1x_1| > 0$. Thus if x_5 lies in or on C' then $L_G > L_{G'}$ for $G' = E \in M$. If x_5 lies outside C' then since s_2 lies inside C', x_5 must lie below l_L. Note also that as it only needs to be shown that G is not minimal at p_o there is now not the restriction of examining networks only from M.

Thus it will be apparent that $|x_1x_5|$ is strictly less than either $|s_1x_1|$ or $|s_2x_5|$. If the former occurs then G is longer than $\{x_2x_3,\ Q(x_3, x_4, x_5),\ x_1x_5\}$, and if the latter occurs then G is longer than $\{x_1x_5,\ Q(x_1, x_2x_3),\ x_3x_4\}$.

This completes the proof of the third case.

Thus since $(T/G) \cup G'$ is a T1 tree and $L_{G'} < L_G$, $|(T/G) \cup G'| < |T|$ and so T cannot be minimal. $\square$

Remark. Note that as the proof was by contradiction method it is clear that the result is best possible. i.e. a T1 network with any angle less than 75° cannot be minimal.

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